UNIVERSITY OF PENNSYLVANIA

TOPOLOGICAL IMMERSION OF PEANIAN CONTINUA IN A SPHERICAL SURFACE

SCHIEFFELIN CLAYTOR

A DISSERTATION

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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TOPOLOGICAL IMMERSION OF PEANIAN CONTINUA IN A SPHERICAL SURFACE[1]

BY SCHIEFFELIN CLAYTOR

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Introduction

In 1930 Kuratowski[2] established the following result:

THEOREM A: *A peanian continuum,*[3] *containing but a finite number of simple closed curves, is homeomorphic with a subset of the plane, provided that it does not contain a topological image of either of the complexes* $\mathfrak{C}$ *and* $\mathfrak{D}$, *where*

$\mathfrak{C}$ *consists of two groups of three vertices each and nine* 1*-cells, in a fashion that each vertex of one group together with each vertex of the other group bounds in a* 1*-cell;*

$\mathfrak{D}$ *consists of five vertices and ten* 1*-cells, in a fashion that each pair of vertices bounds a* 1*-cell.*[4]

This theorem suggests the more general problem treated in this paper, namely, the characterization of the peanian continua which are homeomorphic with a subset of the surface of a sphere.

We find it convenient, first, to designate as a *primitive skew curve* any point-set homeomorphic with either of the complexes $\mathfrak{C}$ and $\mathfrak{D}$; and then, to distinguish among the set of all peanian continua a class, $\mathfrak{R}$, consisting of those which fail to contain a *primitive skew curve.*

Certain boundary sets are defined in an arbitrary continuum, $\mathbf{M}$, of class $\mathfrak{R}$, as follows:

DEFINITION I: *A simple closed curve,* $\mathbf{T}$, *in* $\mathbf{M}$ *is called a boundary curve of* $\mathbf{M}$ *provided that there do not exist in* $\mathbf{M}$-$\mathbf{T}$ *distinct components* E *and* F *such that*

(α) *a point-pair of* $\mathbf{F}(E)$[5] *separates a point pair of* $\mathbf{F}(F)$ *on* $\mathbf{T}$, *or*

(β) $\mathbf{F}(E) \equiv \mathbf{F}(F) = a + b + c$; *where* a, b, c, *are distinct points.*

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[1] Presented to the American Mathematical Society, April 14, 1933.

[2] Sur les problème des courbes gauches en Topologie, Fundamenta Mathematicae, vol. 15, pp. 271–283.

[3] Peanian continuum = a continuous image of a linear interval = a *compact, connected,* and *locally connected* continuum.

[4] A combinatorial equivalent of Theorem A has been given by H. Whitney; Non-separable and Planar Graphs, Trans. Amer. Soc., vol. 34 (1932), pp. 339–362. Also, Mazurkiewicz has recently shown that if M is a one-dimensional peanian continuum which can through no "kleine Abbildungen" be carried into a plane curve, then M necessarily contains a topological image of either $\mathfrak{C}$ or $\mathfrak{D}$. Über nicht plattbare Kurven, Fund. Math., vol. 20, pp. 281–284.

[5] For $L \subset \mathbf{M}$, $\mathbf{F}(L) = \overline{L} \cdot (\overline{M - L})$ = the boundary of L in $\mathbf{M}$.

DEFINITION II: *A simple continuous arc, B, in* $\mathbf{M}$ *is called a boundary arc of* $\mathbf{M}$ *provided that if* $\mathbf{C}$ *is a cyclic element of* $\mathbf{M}$ *containing more than one point of* B, *then the point-set* $B \cdot \mathbf{C}$ *is a subset of some boundary curve of* $\mathbf{C}$. *A point of* $\mathbf{M}$ *is called a boundary point of* $\mathbf{M}$ *provided that it lies on some boundary arc of* $\mathbf{M}$.

We summarize our results:

(A) *If a continuum of class* $\mathfrak{R}$ *is without a boundary curve, then its proper*[6] *cyclic elements are simple closed surfaces.*

(B) *Every continuum of class* $\mathfrak{R}$ *is homeomorphic with a subset of some peanian continuum of which the proper cyclic elements are simple closed surfaces.*

(C) *A necessary and sufficient condition that a continuum,* $\mathbf{M}$, *of class* $\mathfrak{R}$ *be homeomorphic with a subset of a spherical surface is that each cut-point,* P, *of* $\mathbf{M}$ *be a boundary point of the closure of every component of* $\mathbf{M} - P$.

Corollaries:

(a) *A simple closed surface may be characterized as a cyclic continuum of class* $\mathfrak{R}$, *other than a point, which does not have a boundary curve.*

(b) *Every cyclic continuum of class* $\mathfrak{R}$ *is homeomorphic with a subset of a spherical surface.*

(c) *A closed 2-cell may be characterized as a cyclic continuum of class* $\mathfrak{R}$ *that has precisely one boundary curve.*

The result (C) furnishes a complete solution of our central problem, for it is clear that a peanian continuum cannot fail to belong to the class $\mathfrak{R}$ and at the same time have a topological image in the surface of a sphere.

The author takes this opportunity to express his grateful appreciation of the encouragement and counsel that he has received of Professor John R. Kline in the solution of the problems of this paper.

I. Preliminary Lemmas

Throughout our work, $\mathbf{M}$ will be used to denote an arbitrary continuum of class $\mathfrak{R}$; $\mathbf{C}$, to denote an arbitrary cyclic continuum of class $\mathfrak{R}$ that is distinct from a point; and $\mathbf{T}$ to denote any simple closed curve in $\mathbf{M}$ (or in $\mathbf{C}$). A *region* is an *open* and *connected* point-set.

LEMMA 1. *A necessary and sufficient condition that* $\mathbf{T}$ *fail to be a boundary curve of* $\mathbf{M}$ *is that either*

(a) *arcs* ab *and* cd *exist in* $\mathbf{M}$ *having no point in common and only their end points on* $\mathbf{T}$ *such that* $(a + b)$ *separates* $(c + d)$ *on* $\mathbf{T}$; *or if not, then*

(b) *arc triods*[7] T_1 *and* T_2 *exist in* $\mathbf{M}$ *such that* $T_1 \cdot T_2 = T_1 \cdot \mathbf{T} = T_2 \cdot \mathbf{T} = a + b + c$, *where* a, b, c *are the feet of the triods* T_1 *and* T_2.

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[6] A cyclic element of a peanian continuum is said to be *proper* if it consists of more than a single point.

[7] An *arc triod* is a point-set consisting of three simple continuous arcs that have a common end point P which is the only point common to any pair of the arcs. The point P is called the *center* of the triod, while its (three) end points are called its *feet*.

Demonstration. The necessity of our condition is quite obvious, so we consider its sufficiency. There are two cases to be dealt with:

Case 1. Assume that condition (a) holds. Then the segments $\widehat{ab}$[8] and $\widehat{cd}$ do not both lie in a single component D of $\mathbf{M} - \mathbf{T}$; for otherwise, on taking in D an arc ef that has only the end point e on $\widehat{ab}$ and only the end point f on $\widehat{cd}$, it is readily seen that the point-set $\mathbf{T} + ab + cd + ef$ is a *primitive skew curve* (homeomorphic with the complex $\mathfrak{S}$); and this contradicts our assumption that $\mathbf{M}$ is a continuum of class $\mathfrak{R}$. Accordingly, let E and F be the distinct components of $\mathbf{M} - \mathbf{T}$ that contain, respectively, the sets $\widehat{ab}$ and $\widehat{cd}$. It is then clear that the condition (α) of Definition I holds with respect to E and F.

Case 2. Assume that condition (b) holds, while condition (a) *fails* to hold. The treatment of this case varies but slightly from that of the preceding one.

COROLLARY. *If* $\mathbf{T}$ *is a boundary curve of* $\mathbf{M}$ *and* L *is a peanian subcontinuum of* $\mathbf{M}$*that contains* $\mathbf{T}$, *then* $\mathbf{T}$ *is also a boundary curve of* L.

DEFINITIONS. *Distinct components* E *and* F *of* $\mathbf{C} - \mathbf{T}$ *are said to be on opposite sides of* $\mathbf{T}$ *provided* $\mathbf{T}$ *is not a boundary curve of the continuum* $\mathbf{T} + E + F$.

If $E_1, E_2, \cdots, E_n$, $(n > 1)$, *is a finite collection of distinct components of* $\mathbf{C} - \mathbf{T}$ *such that* E_i *and* E_{i+1} *are on opposite sides of* $\mathbf{T}$, $1 \leqq i \leqq n - 1$, *then the point-set* $E_1 + E_2 + \cdots + E_n$ *is called a chain in* $\mathbf{C} - \mathbf{T}$ *joining* E_1 *to* E_n, *and is denoted by* $C[E_1, E_n]$. *The order of a chain is the number of distinct components contained in it.*

A nest, $n(E)$, *determined in* $\mathbf{C} - \mathbf{T}$ *by one of its components* E, *is the point-set consisting of* E *itself together with all the components* E_x *of* $\mathbf{C} - \mathbf{T}$ *for which there exists a chain joining* E *and* E_x.

A few of the properties of *nests* in $\mathbf{C} - \mathbf{T}$ are as follows:

PROPERTY 1. *If* F *is a component of a nest* $n(E)$, *then*

$$n(E) \equiv n(F) .$$

PROPERTY 2. *For an arbitrary pair of nests* $n(E)$ *and* $n(F)$ *we have*

$$n(E) \equiv n(F) \qquad or \qquad n(E) \cdot n(F) = 0 .$$

The demonstrations of Properties 1 and 2 are left to the reader, and we proceed with

PROPERTY 3. *The chains joining two arbitrary components of a nest in* $\mathbf{C} - \mathbf{T}$ *are either all of even order, or else all of odd order.*

Demonstration. On supposing our proposition false, it readily follows that there exist, in $\mathbf{C} - \mathbf{T}$, *chains* $C_1[E, F]$ and $C_2[E, F]$ which are of *even* and *odd* order respectively, and which, furthermore, *have in common only the components* E *and* F. For convenience we let $E_1, E_2, \cdots, E_{2n}$ represent uniquely the components of $C_1[E, F]$ and $E_{2n}, E_{2n+1}, \cdots, E_{2m}$ represent uniquely the components of $C_2[E, F]$, in such a way that E_i and E_{i+1} are on opposite sides of $\mathbf{T}$, for $i = 1, 2, \cdots, 2m - 1$, while $E_1 = E_{2m} = E$ and $E_{2n} = F$.

[8] If ab is a simple continuous arc then ab denotes the point-set $ab - (a + b)$.

It is now a simple matter to choose subsets $e_1, e_2, \cdots, e_{2m}$ of $E_1, E_2, \cdots, E_{2m}$, respectively, so that

(1) $e_1 \equiv e_{2m}$;

(2) e_i is *closed*[9] relative to E_i for $i = 1, 2, \cdots, 2m$;

(3) The closure $\bar{e}_i$ of e_i is, for each value of i, either

i) an arc having only its end points on $\mathbf{T}$, or else

ii) an *arc* triod having only its *feet* on $\mathbf{T}$;

(4) e_i and e_{i+1} are on opposite sides of $\mathbf{T}$ for $i = 1, 2, \cdots, 2m - 1$.

Then, on setting $\Psi = \mathbf{T} + e_1 + e_2 + \cdots + e_{2m}$, we see by (2) and (3) that Ψ is a *peanian* continuum that contains *at most a finite* number of simple closed curves. Furthermore, Ψ is without doubt a continuum of class $\mathfrak{R}$. Hence, by Theorem A,[10] there exist a topological image Ψ' of Ψ in a spherical surface S. Denote by $\mathbf{T}', e_1', e_2', \cdots, e_{2m}'$ the images in Ψ' that correspond respectively to $\mathbf{T}, e_1, e_2, \cdots, e_{2m}$ in Ψ. Then (1) implies

(5) $e_1' \equiv e_{2m}'$, and from (4) it follows that

(6) e_i' and e_{i+1}' are on opposite sides of $\mathbf{T}$ for $i = 1, 2, \cdots, 2m - 1$.

By referring to (6), and the corollary to Lemma 1, it is seen that the components e_i' and e_{i+1}', $(1 \leqq i \leqq 2m - 1)$, of $\Psi' - \mathbf{T}'$ *do not both lie in the same domain complementary to* $\mathbf{T}'$ *in* S. Consequently, on properly designating the two components of $S - \mathbf{T}'$ by I_1 and I_2, we obtain

(7) $I_1 \supset e_1' + e_3' + \cdots + e_{2m-1}' \supset e_1'$, and

(8) $I_2 \supset e_2' + e_4' + \cdots + e_{2m}' \supset e_{2m}'$.

It is now evident that (5), (7), and (8) imply a contradiction, for manifestly the *single* point-set represented by e_1' and e_{2m}' cannot belong to *both* I_1 and I_2. We conclude then that our proposition is valid.

LEMMA 2. *At most a finite number of the components of* $\mathbf{C} - \mathbf{T}$ *are of diameter greater than a given positive quantity.*

Demonstration. Assume, on the contrary, that there exists an infinite sequence

(1) $E_1, E_2, \cdots, E_n, \cdots$ of distinct components E_i of $\mathbf{C} - \mathbf{T}$ for which

(2) $\delta(E_i) > r > 0, \qquad i = 1, 2, \cdots, n, \cdots$.

No generality is lost in supposing that this component sequence converges to a sequential limit L in $\mathbf{C}$; then from (2) we get $\delta(L) > 0$. The continuum L is a subset of the simple closed curve $\mathbf{T}$, and hence is an arc; for, if $u(E_i, \mathbf{T})$ denotes the *upper distance*[11] of E_i from $\mathbf{T}$, then according to a lemma due to Zippin[12] we have

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[9] A subset A of a point-set E is said to be "*closed*" relative to E provided that the limit points of A which belong to E also belong to A.

[10] See Introduction.

[11] The *upper distance* of a point-set M from a point-set N is the upper bound of the numbers $\rho(m, N)$, where m is a point of M.

[12] L. Zippin, "A study of continuous curves and their relation to the Janiszewski-Mullikan Theorem," Trans. Amer. Math. Soc., vol. 31 (1929), pp. 744–770. See lemma of paragraph 3.

$$\lim_{i=\infty} u(E_i, \mathbf{T}) = 0. \tag{3}$$

Now take six distinct points a_0, a_1, a_2, a_3, a_4, a_5 that lie on L, and hence on $\mathbf{T}$, in the order named. Then since $\mathbf{C}$ is locally connected we may choose about a_0, a_1, $\cdots$, a_5 respectivley, regions U_0, U_1, $\cdots$, U_5 of $\mathbf{C}$ such that

(i) $\overline{U}_i \cdot \overline{U}_j = 0$, when $i \neq j$; and

(ii) if x_i is an arbitrary point of $\mathbf{T}$ in U_i, $(0 \leqq i \leqq 5)$, then the point order $x_0 x_1 x_2 x_3 x_4 x_5$ is preserved on $\mathbf{T}$.

By (3), there exists an integer $\bar{n}$ that insures

$$E_n \cdot U_i \neq 0,\ i = 0, 1, \cdots, 5, \text{ for } n > \bar{n}, \tag{4}$$

it being assumed that E_n is an element of the sequence (1). Accordingly, let E_{n_0}, E_{n_1}, E_{n_2} be distinct components of $\mathbf{C} - \mathbf{T}$, appearing in (1), for which

(5) n_0, n_1, n_2 are each $> \bar{n}$; and then put

(6) $W_\sigma = U_\sigma + E_{n_\sigma} + U_{\sigma+3}$, $\sigma = 0, 1, 2$.

Since the U's and E's are *connected open* subsets of $\mathbf{C}$ it follows from (4), (5), and (6) that W_σ is also a *connected open* subset of $\mathbf{C}$. Then from

$$W_\sigma \supset a_\sigma + a_{\sigma+3},\ \sigma = 0, 1, 2,$$

it is clear that, for each value of σ, W_σ contains an arc $a_\sigma a_{\sigma+3}$ joining the point a_σ of U_σ to the point $a_{\sigma+3}$ of $U_{\sigma+3}$. In the light of the relation (i) it is apparent that the arc $a_\sigma a_{\sigma+3}$ has a *last* point b_σ in the point-set $\mathbf{T} \cdot \overline{U}_\sigma$, and *following* b_σ a *first* point $b_{\sigma+3}$ in the point-set $\mathbf{T} \cdot \overline{U}_{\sigma+3}$. Let $b_\sigma b_{\sigma+3}$ denote the subarc on $a_\sigma a_{\sigma+3}$ that joins b_σ to $b_{\sigma+3}$; and set

$$\Pi = \mathbf{T} + b_0 b_3 + b_1 b_4 + b_2 b_5.$$

With the aid of (ii), it is now easily seen that Π is a primitive skew curve homeomorphic with the complex $\mathfrak{C}$. However, $\mathbf{C} \supset \Pi$. Hence we have obtained the contradiction that $\mathbf{C}$ is not a continuum of class $\mathfrak{R}$; consequently, our lemma is proved.

THE NORMAL REPRESENTATION OF $\mathbf{C}$. *If* $\mathbf{C} - \mathbf{T}$ *is resolved into the sum of two of its subsets* A *and* B *in such a way that*

(1) A *and* B *are separated in* $\mathbf{C}$ *by* $\mathbf{T}$, *and*

(2) $\mathbf{T}$ *is a boundary curve of both* $A + \mathbf{T}$ *and* $\mathbf{T} + B$;

then the exhibition of $\mathbf{C}$ *as the sum* $A + \mathbf{T} + B$ *is called a normal representation of* $\mathbf{C}$.

LEMMA 3. *For each simple closed curve* $\mathbf{T}$ *of* $\mathbf{C}$ *there exists a normal representation:* $\mathbf{C} = A + \mathbf{T} + B$.

Demonstration. Consider an arbitrary nest $n(E_\sigma)$ that is determined in

$\mathbf{C} - \mathbf{T}$ by one of its components E_σ. We define a normal representation of the continuum $\mathbf{T} + n(E_\sigma)$ as follows:

(I) Let A_σ consist of E_σ together with those components E_x of $n(E_\sigma)$ that are joined to E_σ by a *chain* in $\mathbf{C} - \mathbf{T}$ of *odd* order.

(II) Let B_σ consist of those components E_y of $n(E_\sigma)$ that are joined to E_σ by a *chain* in $\mathbf{C} - \mathbf{T}$ of *even* order. It is now clear that either of the assumptions

(1) a component of $n(E_\sigma)$ belongs to both A_σ and B_σ,

(2) $\mathbf{T}$ is not a *boundary curve* of both $A_\sigma + \mathbf{T}$ and $\mathbf{T} + B_\sigma$,

will lead directly to a contradiction of the Property 3 relating to nests in $\mathbf{C} - \mathbf{T}$. Consequently we have the normal representation

$$\mathbf{T} + n(E_\sigma) = A_\sigma + \mathbf{T} + B_\sigma . \tag{3}$$

As a result of Lemma 2 it follows that the aggregate of components in $\mathbf{C} - \mathbf{T}$ is at most countably infinite; hence, the set of distinct nests in $\mathbf{C} - \mathbf{T}$ is countable and may be arranged in a simple sequence

$$n(E_1), n(E_2), \cdots, n(E_\sigma), \cdots . \tag{4}$$

Now evidently

$$\mathbf{C} = \mathbf{T} + \sum n(E_\sigma) = \sum [\mathbf{T} + n(E_\sigma)] ;$$

then on referring to relation (3) we get

$$\mathbf{C} = \sum (A_\sigma + \mathbf{T} + B_\sigma) . \tag{5}$$

Finally, set

$$A = A_1 + A_2 + \cdots + A_\sigma + \cdots ,$$

$$B = B_1 + B_2 + \cdots + B_\sigma + \cdots ;$$

and substitute in equation (5) to get

$$\mathbf{C} = A + \mathbf{T} + B ,$$

the desired normal representation.

LEMMA 4. *Let* $\{A_i + \mathbf{T}_i + B_i\}$ *be an infinite sequence of normal representations of* $\mathbf{C}$ *such that*

(1) *the sequence* $\{\mathbf{T}_i\}$ *is equicontinuous*[13] *and converges to a simple closed curve* $\mathbf{T}$,

[13] A collection G of simple closed curves in a metric space is said to be *equicontinuous* provided that for a given positive number ϵ there exists a positive number δ such that if P and Q are points on a curve $\mathfrak{T}$ of G for which $\rho(P, Q) < \delta$, then there is an arc PQ on $\mathfrak{T}$ of diameter $< \epsilon$. Cf. R. L. Moore, *Concerning certain equicontinuous systems of curves*, Trans. Amer. Math. Soc., vol. 22 (1921), p. 42; and by the same author: *On the generation of a simple surface* . . . . , Fund. Math., T. IV, p. 106.

(2) *the sequences* $\{A_i + \mathbf{T}_i\}$ *and* $\{\mathbf{T}_i + B_i\}$ *converge respectively to sequential limits* $A + \mathbf{T}$ *and* $\mathbf{T} + B$, *where neither* A *nor* B *contains a point of* $\mathbf{T}$;

then $A + \mathbf{T} + B$ *is a normal representation of* $\mathbf{C}$.

Demonstration. It is quite evident that $\mathbf{C} - \mathbf{T}$ is the sum of the two point-sets A and B which are *separated* in $\mathbf{C}$ by $\mathbf{T}$. Furthermore, $\mathbf{T}$ is a *boundary curve* of both $A + \mathbf{T}$ and $\mathbf{T} + B$; for on assuming the contrary, we can obtain, without difficulty, the contradiction that $\mathbf{T}_i$ is *not* a *boundary curve* of both $A_i + \mathbf{T}_i$ and $\mathbf{T}_i + B_i$ for some sufficiently large integer i. The details of this discussion are left to the reader, and we proceed with

LEMMA 5. *If* $A_0 + \mathbf{T}_0 + B_0$ *is a normal representation of* $\mathbf{C}$, *and* $\mathbf{T}'$ *a simple closed curve in* $\mathbf{T}_0 + B_0$, *then there exists a normal representation* $\mathbf{C} = A' + \mathbf{T}' + B'$ for which $A' \supset A_0$ and $B' \subset B_0$.

Demonstration. Excluding the trivial case in which $\mathbf{T}' \equiv \mathbf{T}_0$, we assume that $\mathbf{T}' \not\equiv \mathbf{T}_0$, and uniquely arrange the components a_i of $\mathbf{T}' - \mathbf{T}_0$ in a sequence

$$a_1, a_2, \cdots, a_n, \cdots, \tag{1}$$

that may be either finite or infinite. It is now easily shown that to each element a_n of (1) there may be associated a component b_n of $\mathbf{T}_0 - \mathbf{T}'$ such that

(2) $\bar{a}_n$ separates $\mathbf{T}_0 + \mathbf{B}_0$ between b_n and $\mathbf{T}' - \bar{a}_n$ when $\mathbf{T}' - \bar{a}_n \neq 0$; and

(3) $b_i \neq b_j$ when $i \neq j$.

This particular choice of b_n insures that, on setting

$$\mathbf{T}_n = (\mathbf{T}_{n-1} - b_n) + a_n,$$

the point-sets

$$\mathbf{T}_1, \mathbf{T}_2, \cdots, \mathbf{T}_n, \cdots \tag{4}$$

are each simple closed curves. It is furthermore evident that either $\mathbf{T}_n = \mathbf{T}'$ for some value of $n \geqq 1$, or else that $\lim_{n=\infty} \mathbf{T}_n = \mathbf{T}'$, in which case $\{\mathbf{T}_n\}$ is an equicontinuous collection of curves.

With reference to the curve $\mathbf{T}_1$ we now indicate a way of choosing a normal representation

$$\mathbf{C} = A_1 + \mathbf{T}_1 + B_1 \text{ for which } A_1 \supset A_0 \text{ and } B_1 \subset B_0. \tag{5}$$

To this end, consider the nests in $\mathbf{C} - \mathbf{T}_1$. As distinct nests have no points in common (Property 2), there is a unique nest $n(E_\sigma)$ that contains b_1, where the defining component E_σ of $n(E_\sigma)$ is chosen as *that* particular one which contains b_1 (Property 1). Strictly according to the method of Lemma 3, we now effect a normal representation

$$\mathbf{T}_1 + n(E_\sigma) = A_\sigma + \mathbf{T}_1 + B_\sigma \tag{6}$$

with the result that

$$A_\sigma \supset E_\sigma \tag{7}$$

and furthermore that

$$n(E_\sigma)\cdot A_0 = A_\sigma . \tag{8}$$

Let us turn next to the nests $n(E)$ in $\mathbf{C} - \mathbf{T}_1$ that are distinct from $n(E_\sigma)$. These fall into two groups $\sum n(E_\xi)$ and $\sum n(E_\delta)$ (either of which may be vacuous) according as

$$\mathbf{F}\,[n(E)] \subset \mathbf{T}_1 - a_1 \tag{9}$$

or

$$\mathbf{F}\,[n(E)] \subset \bar{a}_1 . \tag{10}$$

A nest $n(E_\xi)$ of $\sum n(E_\xi)$, satisfying relation (9), coincides precisely with a nest in $\mathbf{C} - \mathbf{T}_0$. For each of these set

$$A_\xi = A_0\cdot n(E_\xi), \qquad B_\xi = B_0\cdot n(E_\xi) , \tag{11}$$

and obtain the normal representation

$$\mathbf{T}_1 + n(E_\xi) = A_\xi + \mathbf{T}_1 + B_\xi . \tag{12}$$

A nest $n(E_\delta)$ of $\sum n(E_\delta)$, satisfying relation (10), has no point in A_0, that is,

$$A_0\cdot n(E_\delta) = 0 . \tag{13}$$

For each of these take an arbitrary normal representation

$$\mathbf{T}_1 + n(E_\delta) = A_\delta + \mathbf{T}_1 + B_\delta . \tag{14}$$

Relations (12) and (14) yield

$$\mathbf{T}_1 + \sum n(E_\xi) = \sum A_\xi + \mathbf{T}_1 + \sum B_\xi \tag{15}$$

and

$$\mathbf{T}_1 + \sum n(E_\delta) = \sum A_\delta + \mathbf{T}_1 + \sum B_\delta . \tag{16}$$

Now set

$$A_1 = A_\sigma + \sum A_\xi + \sum A_\delta , \qquad B_1 = B_\sigma + \sum B_\xi + \sum B_\delta ,$$

and combine (6), (15), and (16) to obtain the normal representation (5), where $A_1 \supset A_0$ and $B_1 \subset B_0$ because of (7), (8), (11), and (13).

For the curve $\mathbf{T}_2$ in (4), the condition (2) now aids in establishing $\mathbf{T}_2 \subset \mathbf{T}_1 + B_1$; hence, by proceeding as we did in getting (5), we can construct a normal representation $\mathbf{C} = A_2 + \mathbf{T}_2 + B_2$, where $A_2 \supset A_1$, $B_2 \subset B_1$; furthermore, $\mathbf{T}_3 \subset \mathbf{T}_2 + B_2$.

On continuing in this way indefinitely, we might possibly reach an integer n for which $\mathbf{T}_n = \mathbf{T}'$; then, on putting $A_n = A'$ and $B_n = B'$, it follows that the normal representation $\mathbf{C} = A' + \mathbf{T}' + B'$ satisfies the requirements of our lemma, in virtue of $A_n \supset A_{n-1} \supset \cdots \supset A_1 \supset A_0$ and $B_n \subset B_{n-1} \subset \cdots \subset B_1 \subset B_0$. In case, however, no such integer n exists, then after putting

$$\lim_{n=\infty} (A_n + \mathbf{T}_n) = A' + \mathbf{T}' , \qquad A'\cdot\mathbf{T}' = 0 ,$$

and

$$\lim_{n=-\infty} (\mathbf{T}_n + B_n)' = \mathbf{T}' + B', \qquad \mathbf{T}' \cdot B' = 0,$$

it is easily seen that the sequence of normal representations $\{A_n + \mathbf{T}_n + B_n\}$ satisfies the hypothesis of Lemma 4. Consequently we get the normal representation

$$\mathbf{C} = A' + \mathbf{T}' + B'$$

which fulfills our requirements, since evidently $A' \supset A_0$ and $B' \subset B_0$.

LEMMA 6. *If G is an equicontinuous collection of simple closed curves, all of diameter $\geqq r > 0$, in a compact metric space W, then the closure $\bar{G}$ of G is a self-compact*[14] *equicontinuous collection of simple closed curves.*

Demonstration. Consider, in G, any infinite sequence of curves

$$\mathbf{T}_1, \mathbf{T}_2, \cdots, \mathbf{T}_n, \cdots \tag{1}$$

that converges to a sequential limit L in W. Since, by hypothesis, W is compact and $\mathbf{T}_n$ is of diameter $\geqq r$, it follows that L is a compact continuum. We proceed to show that L is a simple closed curve.

Let a and b be an arbitrary pair of distinct points of L and choose on $\mathbf{T}_n$, for each value of n, points a_n and b_n so that the sequences $\{a_n\}$ and $\{b_n\}$ converge respectively to the points a and b. Denote by A_n and B_n the two arcs determined on $\mathbf{T}_n$ by the point pair (a_n, b_n). Then we can select a subsequence of curves

$$\mathbf{T}_{n1}, \mathbf{T}_{n2}, \cdots, \mathbf{T}_{ni}, \cdots \tag{2}$$

from the sequence (1) such that the corresponding arc sequences $\{A_{ni}\}$ and $\{B_{ni}\}$ converge respectively to sequential limits A and B in L. Clearly $A + B = L$ and $A \cdot B \supset a + b$.

Now suppose that $A \cdot B$ contains a third point c distinct from a and b. Then choose points P_i and Q_i upon A_{ni} and B_{ni} respectively in a fashion that the sequences $\{P_i\}$ and $\{Q_i\}$ both converge to the point c. Next, take ϵ, a positive number, less than both $\rho(a, c)$ and $\rho(b, c)$; then clearly all but a finite number of the arcs $P_i\, a_{ni}\, Q_i$ and $P_i\, b_{ni}\, Q_i$ on $\mathbf{T}_{ni}$ are of diameter $> \epsilon$. Consequently we can choose curves $\mathbf{T}_{nx}$, from (2), for which $\rho(P_x, Q_x)$ is arbitrarily small, while neither of the arcs joining P_x to Q_x in $\mathbf{T}_{nx}$ is of diameter less than ϵ. This, however, implies the contradiction that the collection G is not equicontinuous; hence we conclude that $A \cdot B$ consists of but two points—a and b. Evidently then $L - (a + b)$ is not connected, and since the disconnecting point pair (a, b) was chosen *arbitrarily*, it follows that L is a simple closed curve.

It has been shown that the limit of any convergent sequence of curves of G is a simple closed curve in W. Hence $\bar{G} = G + \{L\}$ is a self-compact collection of curves and is obviously equicontinuous.

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[14] A set G of curves in a compact metric space is said to be *self-compact* if the sequential limit of every convergent subsequence of G is also a curve of G.

II. Continua without Boundary Curves

THEOREM 1. *If a continuum of class* $\mathfrak{R}$ *does not have a boundary curve, then each of its proper cyclic elements is a simple closed surface.*

Demonstration. Assume that $\mathbf{C}$ is such a cyclic element; then according to Zippin,[15] it is sufficient to show that $\mathbf{C}$ satisfies non-vacuously the Jordan Curve Theorem: namely, that every simple closed curve in $\mathbf{C}$ determines in this set precisely two complementary domains of which it is the common boundary.

Let $\mathbf{T}_0$ be an arbitrary simple closed curve in $\mathbf{C}$.[16] Then since $\mathbf{C}$, by hypothesis, has no *boundary curves*, it follows that $\mathbf{C} - \mathbf{T}_0$ consists of more than a single component. Furthermore, there do not exist in $\mathbf{C} - \mathbf{T}_0$ three distinct components each of which has $\mathbf{T}_0$ as its boundary; for, the contrary implies (in contradiction to Lemma 3) that $\mathbf{C}$ does not have a normal representation with reference to $\mathbf{T}_0$. It now remains to be shown that the boundary of each component of $\mathbf{C} - \mathbf{T}_0$ is precisely $\mathbf{T}_0$ itself.

Suppose, on the contrary, that $\mathbf{C} - \mathbf{T}_0$ contains a component D having as its frontier $\mathbf{F}(D)$ a proper subset of $\mathbf{T}_0$. Then on taking a normal representation $\mathbf{C} = A_0 + \mathbf{T}_0 + B_0$, we may, without loss of generality, assume that D is a subset of B_0. Now, if P and Q are a pair of points bounding an interval on $\mathbf{T}_0$ that is complementary to the set $\mathbf{F}(D)$, it readily follows that $\mathbf{T}_0 + B_0$ is *separated* by $P + Q$; consequently, the demonstration of Theorem 1 is effected provided that we establish the following

FUNDAMENTAL LEMMA: *If* $A_0 + \mathbf{T}_0 + B_0$ *is a normal representation of* $\mathbf{C}$ *for which the point-set* $\mathbf{T}_0 + B_0$ *is separated by a point pair* (P, Q) *of* $\mathbf{T}_0$, *then* $\mathbf{C}$ *contains at least one boundary curve* $\mathbf{T}$.

Demonstration. Assume $A_0 + \mathbf{T}_0 + B_0$ to be a normal representation of $\mathbf{C}$ which satisfies our hypothesis. Then with $A_0 + \mathbf{T}_0 + B_0$ as an initial element, we proceed, in an inductive fashion, to obtain an infinite sequence $\mathbf{S}$ of normal representations of $\mathbf{C}$; this will aid us in establishing the existence of a boundary curve $\mathbf{T}$ in $\mathbf{C}$.

Take an infinite sequence of positive numbers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ such that

$$\epsilon_n > 2\epsilon_{n+1}, \qquad n = 1, 2, 3, \cdots. \tag{1}$$

If then the n^{th} element $A_{n-1} + \mathbf{T}_{n-1} + B_{n-1}$ of $\mathbf{S}$ is given, satisfying the condition

(2) $\mathbf{T}_{n-1} + B_{n-1}$ is *separated* by the point-pair (P, Q) of $\mathbf{T}_{n-1}$, we determine the $(n+1)^{\text{st}}$ element of $\mathbf{S}$ as follows:

The continuum $\mathbf{T}_{n-1} + B_{n-1}$, being peanian, can be covered by a finite collection $\mathfrak{S}^n$ of its regions U_i^n, where

$$\delta(U_i^n) < \epsilon_n, \quad \text{for} \quad U_i^n \subset \mathfrak{S}^n. \tag{3}$$

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[15] L. Zippin, On continuous curves and the Jordan Curve Theorem, Amer. Jour. Math., vol. 52 (1930), pp. 331–350. Cf. Theorem 3′, pp. 340–341.

[16] As $\mathfrak{C}$ is cyclic and distinct from a point, it evidently contains at least one simple closed curve.

Let us consider the collection $\mathbf{H}_n$ of normal representations $A_n^\sigma + \mathbf{T}_n^\sigma + B_n^\sigma$ of $\mathbf{C}$, each of which satisfies the conditions

a) $A_n^\sigma \supset A_{n-1}$, $B_n^\sigma \subset B_{n-1}$;

b) $\mathbf{T}_n^\sigma + B_n^\sigma$ is *separated* by the point-pair (P, Q) of $\mathbf{T}$; and

c) B_n^σ does *not* contain any region U_i^n of $\mathfrak{S}^n$.

It is of utmost importance that we be convinced that the collection $\mathbf{H}_n$ is actually non-vacuous. In view of the finiteness of $\mathfrak{S}^n$, this can be shown in an elementary fashion with the aid of Lemma 5. We leave the proof to the reader and proceed to the following

DEFINITION. *If* $\mathbf{T}$ *is a simple closed curve and* ϵ_n *a positive number, then* $\xi_n(\mathbf{T})$ *denotes the greatest integer* r *such that* $\mathbf{T}$ *contains a collection of* r *non-overlapping*[17] *simple continuous arcs, each of diameter* $\geqq \epsilon_n$.

Now set

(4) $\pi_n =$ lower bound $\xi_{n-1}(\mathbf{T}_n^\sigma)$, for $A_n^\sigma + \mathbf{T}_n^\sigma + B_n^\sigma \subset \mathbf{H}_n$;

and choose $A_n + \mathbf{T}_n + B_n$ an arbitrary normal representation of $\mathbf{H}_n$ for which $\xi_{n-1}(\mathbf{T}_n) = \pi_n$. We take the representation $A_n + \mathbf{T}_n + B_n$ as the $(n+1)^{\text{st}}$ element of the collection $\mathbf{S}$.

With the initial element $A_0 + \mathbf{T}_0 + B_0$ having been given, and in addition, a method for obtaining an $(n+1)^{\text{st}}$ normal representation when the n^{th} is determined, we generate an infinite sequence of normal representations $\mathbf{S} \equiv \{A_n + \mathbf{T}_n + B_n\}$ having the following properties:

A) $A_n \supset A_{n-1}$, $B_n \subset B_{n-1}$, $n \geqq 1$;

B) $\mathbf{T}_n + B_n$ is *separated* by the point-pair (P, Q) of $\mathbf{T}_n$, $n \geqq 0$;

C) B_n does *not* contain any region U_i^n of $\mathfrak{S}^n$, $n \geqq 1$;

D) $\xi_{n-1}(\mathbf{T}_n) = \pi_n$, $n \geqq 1$.

Let $\{\mathbf{T}_n\}$ be the sequence of separating curves associated with the normal representations of $\mathbf{S}$. Then, for integers $l > m > n$, it follows, as a consequence of A), that

(5) $\mathbf{T}_m$ *separates* the point-sets $(\mathbf{T}_l - \mathbf{T}_m)$ and $(\mathbf{T}_n - \mathbf{T}_m)$ in $\mathbf{C}$.

We establish two properties of $\{\mathbf{T}_n\}$:

PROPERTY I: *The sequence* $\{\mathbf{T}_n\}$ *converges to a sequential limit.* For suppose $\{\mathbf{T}_{u_i}\}$ and $\{\mathbf{T}_{v_i}\}$ are subsequences of $\{\mathbf{T}_n\}$ that converge respectively to distinct sequential limits L and M in $\mathbf{C}$. Then one of the sets L and M, say M, contains a point T that does not belong to L. Take a region $U(T)$ in $\mathbf{C}$ of diameter $< \frac{1}{2}\rho(T, L)$ and let $\bar{n}$ be an integer such that for $v_i > \bar{n}$, $\mathbf{T}_{v_i}$ has points in $U(T)$.

[17] A collection of arcs L is said to be *non-overlapping* provided that no interior point of an arc of L belongs to any other arc of L.

Select also an integer $\bar{m}$ such that for $u_i > \bar{m}$ the upper distance of $\mathbf{T}_{u_i}$ from L is less than $\frac{1}{2}\rho(T, L)$; so that $\mathbf{T}_{u_i}$ has *no* point in $U(T)$. Now choose three integers u_a, u_b, u_c satisfying $u_a < \bar{m}$, $v_b > \bar{n} < v_c$, $v_b < u_a < v_c$.

It appears then that $\mathbf{T}_{u_a}$ does not separate $\mathbf{C}$ between $(\mathbf{T}_{v_b} - \mathbf{T}_{u_c})$ and $(\mathbf{T}_{v_c} - \mathbf{T}_{u_a})$ since both these sets have points in the region $U(T)$, while $\mathbf{T}_{u_a} \cdot U(T) = 0$. We have then a contradiction of 5); and Property I is established.

Property II. *The curves of* $\{\mathbf{T}_n\}$ *form an equicontinuous collection.* Suppose that this is not so; then it readily follows that there exists an infinite subsequence $\{\mathbf{T}_{n_i}\}$ of $\{\mathbf{T}_n\}$, and in $\mathbf{C}$, an infinite sequence of arcs $\{a_i b_i\}$ such that

1°) $n_i < n_{i+1}$, $\quad i = 1, 2, 3, \cdots$;

2°) $(a_i b_i) \cdot \mathbf{T}_{n_i} = a_i + b_i$, $\quad i = 1, 2, 3, \cdots$;

3°) $\lim_{i=\infty} a_i b_i =$ a point T in $\mathbf{C}$; and

4°) The two arcs joining the points a_i and b_i on $\mathbf{T}_{n_i}$ can be denoted by $E^1_{n_i}$ and $E^2_{n_i}$ in such a way that the arc sequences $\{E^1_{n_i}\}$ and $\{E^2_{n_i}\}$ converge respectively to sequential limiting continua E^1 and E^2, each of diameter > 0.

This situation suggests a consideration of the following possibilities:

Case 1. The sequences $\{\mathbf{T}_{n_i}\}$ and $\{a_i b_i\}$ were so chosen as to satisfy the condition

5°) $\qquad \overset{\frown}{a_i b_i} \subset B_{n_i}, \qquad i = 1, 2, 3, \cdots$.

Since 1), the continuum $\mathbf{C}$ is cyclic, and 2), the continua E^1 and E^2 are of diameter > 0, it follows that a region U of $\mathbf{C}$ can be taken about the point T such that

(6) $\mathbf{C} - \overline{U}$ is a *connected* point-set, and

(7) $\delta(U) < \delta(E^u)$, $u = 1, 2$. As a consequence of 4°) and 7), there is an integer v_1 such that

(8) $(\mathbf{C} - \overline{U}) \cdot E^u_{n_v} \neq 0$, $\quad u = 1, 2,$ $\quad$ when $v > v_1$.

For each integer h, let W_h denote the subcollection of $\mathfrak{S}^{n_h}$ which consists of all of those regions $U_i^{n_h}$ for which

(9) $(a_h b_h) \cdot U_i^{n_h} \neq 0$.

Then, by 1), 3), and 3°) there exists an integer v_2 such that

(10) $\rho(U_i^{n_v}, \mathbf{C} - \overline{U}) > 3\epsilon_{n_v - 1}$, when $U_i^{n_v} \subset W_v$ and $v > v_2$.

Now let v assume a fixed value greater than both v_1 and v_2, so that the relations 8) and 10) hold simultaneously. It readily follows from 6) and 8) that

the point-set $\mathbf{C} - \overline{U}$ contains an arc e_1e_2 that joins a point e_1 of $E^1_{n_v}$ to a point e_2 of $E^2_{n_v}$, and which has no other point on the curve $\mathbf{T}_{n_v} \equiv E^1_{n_v} + E^2_{n_v}$.

In consequence of C), 5°), and 9) it is evident that every region of the collection W_v must contain at least one point of the curve $\mathbf{T}_{n_v}$. Hence, since $\mathbf{T}_{n_v}$ is the sum of the two arcs $e_1a_ve_2$ and $e_1b_ve_2$, it follows that if $U^{n_v}_i$ is any region of W_v, then either

(11) $e_1a_ve_2 \cdot U^{n_v}_i \neq 0$, or else

(12) $e_1b_ve_2 \cdot U^{n_v}_i \neq 0$.

Accordingly, we have the representation $W_v = X + Y$, where a region of W_v belongs to X provided it satisfies the relation 11), and belongs to Y provided it satisfies the relation 12).

Since every point of the arc a_vb_v lies in some region of the collection W_v, it is evident that the point-set consisting of the sum of the points of all the regions of W_v is *connected*. Hence there is some region $U^{n_v}_x$ of X that contains a point of (or is possibly identical with) a region $U^{n_v}_y$ of Y. Consider the point-set

(13) $V = U^{n_v}_x + U^{n_v}_y$ which is *open* and *connected*, and consequently a region. From 1), 3), and 13) we get

(14) $$\delta(V) < 2\epsilon_{n_v} < \epsilon_{n_v-1}.$$

There evidently exists in V an arc ab that joins a point a on arc $e_1a_ve_2$ to a point b on arc $e_1b_ve_2$, without having any other point on the curve $\mathbf{T}_{n_v}$. Then

(15) $\overset{\frown}{ab} \subset B_{n_v}$; for, since the point-pairs (a_v, b_v) and (e_1, e_2) separate each other on the *boundary curve* $\mathbf{T}_{n_v}$ of $(\mathbf{T}_{n_v} + B_{n_v})$, it follows from relation 5⁰) and Lemma 1 that $\overset{\frown}{e_1e_2} \subset A_{n_v}$; consequently, since also the point-pairs (a, b) and (e_1, e_2) separate each other on $\mathbf{T}_{n_v}$, we conclude in an analogous fashion that $\overset{\frown}{ab} \subset B_{n_v}$.

Let us consider the two arcs ae_1b and ae_2b upon $\mathbf{T}_{n_v}$. Since, by B), the point-set $\mathbf{T}_{n_v} + B_{n_v}$ is separated by the point-pair (P, Q) of $\mathbf{T}_{n_v}$, it may readily be shown, with the aid of 15), that one of the arcs (ae_1b, ae_2b), say ae_1b, is such that if

(16) $\mathbf{T}^\sigma_{n_v} = ab + ae_1b$, then P and Q are points of $\mathbf{T}^\sigma_{n_v}$, and

(17) $\mathbf{T}^\sigma_{n_v} - (P + Q)$ is *not* a subset of a *single* component of $\mathbf{T}_{n_v} + B_{n_v} - (P + Q)$.

Now, since $\mathbf{T}^\sigma_{n_v} \subset \mathbf{T}_{n_v} + B_{n_v}$, there exists, according to Lemma 5, a normal representation

(18) $\mathbf{C} = A^\sigma_{n_v} + \mathbf{T}^\sigma_{n_v} + B^\sigma_{n_v}$, where $A^\sigma_{n_v} \supset A_{n_v}$, and $B^\sigma_{n_v} \subset B_{n_v}$.

The relation A), with n replaced by n_v, yields $A_{n_v} \supset A_{n_{v-1}}$ and $B_{n_v} \supset B_{n_{v-1}}$, so that, by 18) we get

a′) $A^{\sigma}_{n_v} \supset A_{n_{v-1}}$, $B^{\sigma}_{n_v} \subset B_{n_{v-1}}$. Also, from $\mathbf{T}^{\sigma}_{n_v} + B^{\sigma}_{n_v} \subset \mathbf{T}_{n_v} + B_{n_v}$ and 17) we observe that $\mathbf{T}^{\sigma}_{n_v} - (P + Q)$ is *not* a subset of a *single* component of $\mathbf{T}^{\sigma}_{n_v} + B^{\sigma}_{n_v} - (P + Q)$, which implies that

b′) $\mathbf{T}^{\sigma}_{n_v} + B^{\sigma}_{n_v}$ is *separated* by the point-pair (P, Q) of $\mathbf{T}^{\sigma}_{n_v}$.

Furthermore, the relation c), together with $B^{\sigma}_{n_v} \subset B_{n_v}$, insures that

c′) $B^{\sigma}_{n_v}$ does *not* contain any region $U^{n_v}_i$ of $\mathfrak{S}^{n_v}$.

In view of a′), b′), and c′) we conclude that the normal representation

$$A^{\sigma}_{n_v} + \mathbf{T}^{\sigma}_{n_v} + B^{\sigma}_{n_v} \text{ is a member of the collection } \mathbf{H}_{n_v}.$$

Now let L be a collection consisting of $\xi_{n_{v-1}}(\mathbf{T}^{\sigma}_{n_v})$ non-overlapping arcs, of diameter $\geqq \epsilon_{n_{v-1}}$, lying in the curve $\mathbf{T}^{\sigma}_{n_v}$. From 14) and $ab \subset V$ we get $\delta(ab) < \epsilon_{n_{v-1}}$.

Consequently, it appears that at *most* 2 of the arcs of L have points upon the arc ab, implying, by 16), that at *least* $\xi_{n_{v-1}}(\mathbf{T}^{\sigma}_{n_v}) - 2$ of the arcs of L lie within arc ae_1b. Since, however, a and b are points of the region V, while e_2 is a point of $\mathbf{C} - \overline{U}$, it follows from 10) and 13) that ae_2b is an arc of diameter $> 3\,\epsilon_{n_{v-1}}$, implying that ae_2b contains a set consisting of at *least* 3 non-overlapping simply continuous arcs. Then, from $\mathbf{T}_{n_v} = ae_1b + ae_2b$, we see finally that $\xi_{n_{v-1}}(\mathbf{T}_{n_v}) \geqq [\xi_{n_{v-1}}(\mathbf{T}^{\sigma}_{n_v}) - 2] + 3 = \xi_{n_{v-1}}(\mathbf{T}^{\sigma}_{n_v}) + 1$, which, in view of the relations 4) and $A^{\sigma}_{n_v} + \mathbf{T}^{\sigma}_{n_v} + B^{\sigma}_{n_v} \subset \mathbf{H}_{n_v}$, is a contradiction of $\xi_{n-1}(\mathbf{T}_n) = \pi_n$, for $n = n_v$. This contradiction disposes of Case 1, and we turn to

Case 2: It is *impossible* to choose the sequences $\{\mathbf{T}_{n_i}\}$ and $\{a_ib_i\}$ so as to satisfy the condition 5°) of Case 1.

Then we may assume, without loss of generality, that $\{\mathbf{T}_{n_i}\}$ and $\{a_ib_i\}$ were so chosen that

$$\overset{\frown}{a_ib_i} \subset A_{n_i}, \qquad i = 1, 2, 3, \cdots. \tag{19}$$

As a further consequence of our hypothesis it can be shown that the two continua E^1 and E^2 have but *one* common point, namely the point T. Take T_μ an arbitrary point of $E^\mu - T$, $u = 1, 2$; and about T_μ let U_μ be a region of $\mathbf{C}$ such that

$$\overline{U}_1 \cdot E^2 = \overline{U}_2 \cdot E^1 = 0. \tag{20}$$

In view of 3°), 4°), and 20), there exists an integer k such that for $r > k$ we have with reference to the simple closed curve $\mathbf{T}_{n_r} = E^1_{n_r} + E^2_{n_r}$, the relations

$$\mathbf{T}_{n_r} \cdot U_\mu = E^\mu_{n_r} \cdot U_\mu \neq 0, \qquad \mu = 1, 2, \text{ and} \tag{21}$$

(22) $a_r b_r \subset \mathbf{C} - (\overline{U}_1 + \overline{U}_2)$. Let e_1 and e_2 be points of the curve $\mathbf{T}_{nk}$ that ie, respectively, in the regions U_1 and U_2. Then

$$\mathbf{T}_{nk} = e_1 a_k e_2 + e_1 b_k e_2$$

and for each integer $x > k$ we have

(23) $$e_1 a_k e_2 \cdot a_x b_x \neq 0 \neq e_1 b_k e_2 \cdot a_x b_x, \text{ where } a_x b_x \subset \{a_i b_i\};$$

for, if either of the arcs $e_1 a_k e_2$ and $e_1 b_k e_2$ should *fail* to contain a point of arc $a_x b_x$ when $x > k$, then in virtue of $\mathbf{T}_{nk} \subset A_{nx} + \mathbf{T}_{nx}$, 20), 21), and 22), it would follow that an arc $c_x d_x$ can be determined in $A_{nx} + \mathbf{T}_{nx}$, having no point on arc $a_x b_x$ and only its end points on $\mathbf{T}_{nx}$, in a fashion that the point pairs (a_x, b_x) and (c_x, d_x) separate each other on $\mathbf{T}_{nx}$. Then, by 19) and condition (a) of Lemma 1, we arrive at the contradiction that $\mathbf{T}_{nx}$ is not a *boundary curve* of the continuum $A_{nx} + \mathbf{T}_{nx}$.

As a consequence of the validity of 23) for all integers $x > k$ we observe, from $\lim_{i=\infty} a_i b_i = T$, (see 3°), that T is a point of each of the arcs $e_1 a_k e_2$ and $e_1 b_k e_2$. However, the relations $U_1 \supset e_1$, $U_2 \supset e_2$, $E^1 \supset T \subset E^2$, and 20) imply that the point T is an *interior* point of each of these arcs. We are, therefore, confronted with the contradiction that the simple closed curve $\mathbf{T}_{nk} = e_1 a_k e_2 + e_1 b_k e_2$ has a *singular* point, namely, the point T.

The contradictions that have been obtained in the treatment of Cases 1 and 2 establish, without doubt, the *equicontinuity* of the collection $\{\mathbf{T}_n\}$.

In virtue of Properties I and II, it follows, by Lemma 6, that the sequential limit of $\{\mathbf{T}_n\}$ is a simple closed curve $\mathbf{T}$ of $\mathbf{C}$. Also, the relation A) insures that the sequences $\{A_n + \mathbf{T}_n\}$ and $\{\mathbf{T}_n + B_n\}$ both converge to *sequential limits* which we denote respectively by $A + \mathbf{T}$ and $\mathbf{T} + B$, under the assumption that $A \cdot \mathbf{T} = \mathbf{T} \cdot B = 0$. By Lemma 4, it follows then that $A + \mathbf{T} + B$ is a normal representation of $\mathbf{C}$. Furthermore, the point-set B is *vacuous*, for it can be easily shown that $B \neq 0$ implies a contradiction of the condition C) for a *sufficiently large* value of n. Consequently, we have the normal representation $\mathbf{C} = A + \mathbf{T}$, which establishes $\mathbf{T}$ as a *boundary curve* of $\mathbf{C}$. This completes the proof of our Fundamental Lemma.

As a direct consequence of Theorem 1 we get the following

COROLLARY. *A simple closed surface may be characterized as a cyclic continuum of class* $\mathfrak{R}$, *distinct from a point, which does not have a boundary curve.*

III. The Boundary Curves of $\mathbf{M}_1$

We prove two theorems concerning these curves:

THEOREM 2. *The collection* G^r, *consisting of the boundary curves in* $\mathbf{M}$ *of diameter* $\geqq r$, *is a self-compact equicontinuous collection.*

Demonstration. We show first that G^r is equicontinuous. Suppose that this is not so; then it follows that G^r contains an infinite sequence of curves

(1) $$\mathbf{T}_1, \mathbf{T}_2, \cdots, \mathbf{T}_n, \cdots$$

that contain, respectively, point pairs

$$(P_1, Q_1), (P_2, Q_2), \cdots, (P_n, Q_n), \cdots$$

such that

(i) $\lim_{n=\infty} \rho(P_n, Q_n) = 0$, and

(ii) both arcs joining P_n to Q_n on $\mathbf{T}_n$ are of diameter $> \epsilon > 0$.

Let A_n and B_n denote the two arcs determined on $\mathbf{T}_n$ by the point pair (P_n, Q_n); then we can choose from the sequence (1) a subsequence $\{\mathbf{T}_{n_i}\}$ such that the corresponding sequences $\{A_{ni}\}$, $\{B_{ni}\}$, and $\{P_{ni}\}$ converge respectively to sequential limits A, B, and T. The relation (i) now implies that the sequence $\{Q_{n_i}\}$ also converges to T. The relation (ii) implies, furthermore, that the continua A and B are both of diameter > 0, and hence contain, respectively, points a and b each distinct from the point T. Now select, for each value of i, a point a_i on A_{n_i} and a point b_i on B_{n_i}, so that the sequence $\{a_i\}$ converges to the point a, while the sequence $\{b_i\}$ converges to the point b.

Clearly all but a finite number of the curves of $\{\mathbf{T}_{n_i}\}$ are a subset of a single cyclic element $\mathbf{C}$ of $\mathbf{M}$; so that the points a, b, and T belong to $\mathbf{C}$. Take in $\mathbf{C}$ a region R containing T for which $\mathbf{C} - \bar{R} \supset a + b$. Then, as T is not a cut-point of $\mathbf{C}$, there exists about T a sub-region U of R such that $\mathbf{C} - U$ is a *peanian* continuum.[18] Since all but a finite number of the point pairs (a_i, b_i) lie in $\mathbf{C} - U$, and all but a finite number of the point pairs (P_{n_i}, Q_{n_i}) lie in U, it follows that for a sufficiently large integer v, we get

$$a_v + b_v \subset \mathbf{C} - U \quad \text{and} \quad P_{n_v} + Q_{n_v} \subset U .$$

In the peanian continuum $\mathbf{C} - U$ take an arc $a_v b_v$ joining a_v to b_v; and in the region U, take an arc $P_{n_v} Q_{n_v}$ joining P_{n_v} to Q_{n_v}. The arc $a_v b_v$ clearly contains a sub-arc $\alpha\beta$ that joins a point α of A_{n_v} to a point β of B_{n_v} and which has no other points on $\mathbf{T}_{n_v}$.

Now consider the two arcs $\alpha P_{n_v}\beta$ and $\alpha Q_{n_v}\beta$ that are determined on $\mathbf{T}_{n_v}$ by the point pair (α, β). The arc $P_{n_v} Q_{n_v}$ contains a sub-arc $\gamma\delta$ that joins a point γ on $\alpha P_{n_v}\beta$ to a point δ on $\alpha Q_{n_v}\beta$ and which has no other points on $\mathbf{T}_{n_v}$.

The two arcs $\alpha\beta$ and $\gamma\delta$ evidently have no common points, and furthermore, the point pairs (α, β) and (γ, δ) separate each other on $\mathbf{T}_{n_v}$. Consequently it follows from condition (a) of Lemma 1 that $\mathbf{T}_{n_v}$ is not a *boundary curve* of $\mathbf{M}$; and in view of this contradiction we conclude that the collection G^r must be equicontinuous.

It is now shown that G^r is a self-compact collection of curves. Consider an arbitrary sequence $\{\mathbf{T}_n\}$ of curves of G^r that converges to a sequential limiting

[18] Cf. C. Kuratowski, Une caractérisation topologique de la surface de la sphère, Fund. Math., vol. 13 (1929), pp. 307–318; see *property* (γ), pp. 314–315; and note 2), p. 315.

set in $\mathbf{M}$; then it follows, from Lemma 6, that this limiting set is a simple closed curve $\mathbf{T}$. We proceed to identify $\mathbf{T}$ as a boundary curve of $\mathbf{M}$.

Let $\mathbf{C}$ be the cyclic element of $\mathbf{M}$ that contains $\mathbf{T}$; then there is evidently an integer μ such that $\mathbf{T}_n \subset \mathbf{C}$ for $n > \mu$. Furthermore, since $\mathbf{T}_n$, $n > \mu$, is, by hypothesis, a boundary curve of $\mathbf{M}$, and hence of $\mathbf{C}$, we clearly have the normal representation $\mathbf{C} = A_n + \mathbf{T}_n$. Now consider the infinite sequence of normal representations $\{A_n + \mathbf{T}_n\}_{n>\mu}$ of $\mathbf{C}$. This sequence evidently satisfies the hypothesis of Lemma 4; consequently, on setting

$$\lim_{n=\infty} A_n + \mathbf{T}_n = A + \mathbf{T},$$

where $A \cdot \mathbf{T} = 0$, we get the normal representation $\mathbf{C} = A + \mathbf{T}$. This implies that $\mathbf{T}$ is a *boundary curve* of $\mathbf{C}$, and hence of $\mathbf{M}$ (since $\mathbf{C}$ is a cyclic element of $\mathbf{M}$). It now follows, by definition, that G^r is self-compact.

THEOREM 3. *If $\{\mathbf{T}_n\}$ is a sequence of boundary curves of $\mathbf{M}$ that converges to a boundary curve $\mathbf{T}$ as a sequential limit, then the set of points, each of which is common to all but a finite number of the curves of $\{\mathbf{T}_n\}$, is dense on $\mathbf{T}$.*

Demonstration. We see from Theorem 2 that the sequence $\{\mathbf{T}_n\}$ is equicontinuous. This, with the aid of our hypothesis that $\mathbf{T}$ and $\mathbf{T}_n$, $n = 1, 2, \cdots$, are *boundary curves*, easily leads to the conclusion that if D is any component of $\mathbf{C} - \mathbf{T}$, then there exists an integer n_D such that

$$\mathbf{F}(D) \subset \mathbf{T}_n, \quad \text{for} \quad n > n_D. \tag{1}$$

If, furthermore, B is any segment on $\mathbf{T}$ that contains no point of $\mathbf{F}(\mathbf{C} - \mathbf{T})$ then it is clear that there exists an integer n_B such that

$$B \subset \mathbf{T}_n, \qquad \text{for} \qquad n > n_B.$$

Now let L be an arbitrary segment on $\mathbf{T}$. If for some component D of $\mathbf{C} - \mathbf{T}$ we have $\mathbf{F}(D) \cdot L \neq 0$, then it follows, from (1), that the segment L contains at least one point that is common to all but a finite number of the curves $\{\mathbf{T}_n\}$. If, however, for each component D of $\mathbf{C} - \mathbf{T}$, we have

$$\mathbf{F}(D) \cdot L = 0, \tag{2}$$

then, since L is an *open* subset of $\mathbf{T}$, it follows from Lemma 2 that relation (2) implies $\mathbf{F}(\mathbf{C} - \mathbf{T}) \cdot L = 0$. We conclude then that L is actually contained in all but a finite number of the curves $\{\mathbf{T}_n\}$. It has now been shown that an *arbitrary* segment L on $\mathbf{T}$ contains points common to all but a finite number of the curves $\{\mathbf{T}_n\}$; hence the set of such points is *dense* on $\mathbf{T}$.

IV. The Extension of the Property of Belonging to Class $\mathfrak{R}$

THEOREM E_1. *Let $C, E_1, E_2, \cdots, E_n, \cdots$, be an infinite sequence of peanian continua such that*

(a) $C_n = C + \sum_{i=1}^{n} E_i$, *is a continuum of class* $\mathfrak{R}$, $\}$ $n = 1, 2, \cdots$,

(b) $C \cdot E_n \neq 0$,

(c) $\lim_{n=\infty} \delta(E_n) = 0$;

then $C_\omega = C + \sum_{i=1}^{\infty} E_i$ *is a continuum of class* $\mathfrak{R}$.

Demonstration. Let ϵ be an arbitrary positive number. Then, according to condition (c) of our hypothesis, there exists an integer r such that

$$\delta(E_\mu) < \epsilon/3, \qquad \text{for} \qquad \mu > r. \tag{1}$$

Since the continuum C_r is peanian, it can be represented as the sum of a finite number of its peanian subcontinua

$$L_1^r, L_2^r, \cdots, L_s^r,$$

where

$$\delta(L_i^r) < \epsilon/3, \qquad i = 1, 2, \cdots, s. \tag{2}$$

Let us consider the continuum C_μ for $\mu > r$. Evidently

$$C_\mu = L_1^r + L_2^r + \cdots + L_s^r + E_{r+1} + \cdots + E_\mu. \tag{3}$$

Now let $H_i^{r,\mu}$, $(1 \leqq i \leqq s)$, denote the sum of those continua among $E_{r+1}, \cdots, E_\mu$ each of which has a point in common with L_i^r. Then, on setting

$$L_i^\mu = L_i^r + H_i^{r,\mu}, \qquad i = 1, 2, \cdots, s, \tag{4}$$

we see from (3) and (4) that

$$C_\mu = L_1^\mu + L_2^\mu + \cdots + L_s^\mu, \qquad \mu > r, \tag{5}$$

where the continua L_i^μ are evidently *peanian*; and

$$\delta(L_i^\mu) < \epsilon, \qquad i = 1, 2, \cdots, s, \tag{6}$$

because of (1), (2), and (4).

Let us now put

$$L_i^\omega = \sum_{\mu=r+1}^{\infty} L_i^\mu, \qquad i = 1, 2, \cdots, s;$$

then evidently

$$C_\omega = L_1^\omega + L_2^\omega + \cdots + L_s^\omega, \tag{7}$$

where again by (1), (2), and (4) we have

$$\delta(L_i^\omega) < \epsilon, \qquad i = 1, 2, \cdots, s. \tag{8}$$

The relations (7) and (8) imply that the obviously compact point-set C_ω is in fact a *peanian* continuum. By a precisely similar argument it follows that L_i^ω, $(1 \leqq i \leqq s)$, is also a peanian continuum.

We now show that the continuum C_ω does not contain a *primitive skew curve*.

Suppose, on the contrary, that C_ω does contain such a curve Π having vertices $P_1, P_2, P_3, \cdots$ and edges $P_1P_2, P_1P_3, P_1P_4, \cdots$. Then about the respective vertices $P_1, P_2, P_3, \cdots$ choose regions $U_1, U_2, U_3, \cdots$ in C_ω such that

i) $\rho(U_x, U_y) > 0$ for each pair of distinct regions, and

ii) $\overline{U}_x$ contains no point on any edge of Π not on the vertex P_x (for each value of x).

We now assume that the positive number ϵ (which was introduced at the beginning of this demonstration) is subject to the following restrictions

(9) $\epsilon < \frac{1}{2}\rho(U_x, U_y)$ for each pair of distinct regions,

(10) the region U_x contains all points in C_ω at a distance $< \epsilon$ from P_x (for each value of x),

(11) if p and q are points of $\Pi - \sum U_x$ that lie respectively on *distinct* edges of Π, then $\epsilon < \frac{1}{2}\rho(p, q)$.

For each edge P_xP_y of Π, let $T^\omega_{x,y}$ denote the point-set consisting of those continua L^ω_i, appearing in (7), that have a point in common with P_xP_y. The set $T^\omega_{x,y}$ is evidently a peanian continuum containing the two vertices P_x and P_y. If the edges P_iP_j and P_kP_l do not have a common vertex, then, from (8), (9), and (11), we get

$$T^\omega_{i,j} \cdot T^\omega_{k,l} = 0, \tag{12}$$

while if P_iP_j and P_jP_k have just one common vertex, then, from (8), (9), (10), and (11), we get

$$P_j \subset T^\omega_{i,j} \cdot T^\omega_{j,k} \subset U_j. \tag{13}$$

Now choose an integer $m \geqq r$ in a fashion that

(14) every vertex P_x of Π is contained in the continuum C_m;

and for each integer $\nu \geqq m$ let $T^\nu_{x,y}$ denote the point-set consisting of those continua L^ν_i appearing in (5), when μ is replaced by ν, that have a point in common with the edge P_xP_y of Π. In general the closed point-set $T^\nu_{x,y}$ *will not be connected* for all values of $\nu \geqq m$; however, with the aid of condition (b) of our hypothesis, it follows that an integer $m_{x,y}$ exists such that

$$T^\nu_{x,y} \text{ is } \textit{connected} \text{ for } \nu \geqq m_{x,y}; \tag{15}$$

(to assume that $m_{x,y}$ does not exist will lead to the contradiction that $T^\omega_{x,y}$ is not connected). Now let $\bar{m}$ be the greatest of the integers $m_{x,y}$ associated with the edges P_xP_y of Π. Then each of the point-sets $T^m_{x,y}$ is *connected*, and, moreover, is a *peanian* continuum (since each continuum L^m_i is peanian).

Since $T^{\bar{m}}_{x,y} \subset T^\omega_{x,y}$, the relation (12) implies

$$T^{\bar{m}}_{i,j} \cdot T^m_{k,l} = 0, \qquad \text{when} \qquad P_iP_j \cdot P_kP_l = 0; \tag{16}$$

while, from (13) and (14), we get

$$(17) \qquad P_i \subset T^{\bar{m}}_{i,j} \cdot T^{\bar{m}}_{j,k} \subset U_j, \qquad \text{when} \qquad P_i P_j \cdot P_j P_k = P_j .$$

In each continuum $T^{\bar{m}}_{x,y}$ let us take, finally, an arc $t^{\bar{m}}_{x,y}$ joining P_x to P_y. Then consider the point-set $\Pi_{\bar{m}} = \sum t^{\bar{m}}_{x,y}$. With the aid of (16) and (17), it readily follows that $\Pi_{\bar{m}}$ contains a primitive skew curve Ω.[19] Furthermore, Ω is evidently a subset of $C_{\bar{m}}$. This conclusion is clearly a contradiction of condition (a) of our hypothesis; therefore our theorem is proved.

THEOREM E_2. *If* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are continua of class* $\mathfrak{R}$ *such that* $\mathbf{M}_1 \cdot \mathbf{M}_2$ *is a boundary curve of both* $\mathbf{M}_1$ *and* $\mathbf{M}_2$, *then* $\mathbf{M}_1 + \mathbf{M}_2$ *is a continuum of class* $\mathfrak{R}$.

Demonstration. We assume our proposition false, implying then that the peanian continuum $\mathbf{M}_1 + \mathbf{M}_2$ contains a *primitive skew curve* Π. Let $\mathbf{T}$ denote the simple closed curve $\mathbf{M}_1 \cdot \mathbf{M}_2$, and consider the point-set $\Pi - \mathbf{T}$:

Case 1: $\Pi - \mathbf{T}$ consists of but a *finite* number of components. It readily follows then that the continuum $\Pi + \mathbf{T}$ contains at *most* a finite number of simple closed curves, and is in fact a *linear graph.*

Putting

$$\Gamma_\mu = \Pi \cdot \mathbf{M}_\mu + \mathbf{T}, \qquad \mu = 1, 2,$$

we observe that Γ_1 and Γ_2 are each linear graphs having $\mathbf{T}$ as a *boundary curve* and *failing* to contain a *primitive skew curve.* Consequently, by Theorem A, there exists, for $\mu = 1, 2$, a topological transformation $t_\mu(P)$ carrying Γ_μ into a subset $\Gamma'_\mu = t_\mu(\Gamma_\mu)$ of a spherical surface S. The image $t_\mu(\mathbf{T})$ of $\mathbf{T}$ in Γ'_μ is denoted by $\mathbf{T}'_\mu$, $\mu = 1, 2$.

At this point we introduce a plausible proposition on linear graphs, the proof of which we omit.

PROPOSITION K. *If* H *is a linear graph lying in a spherical surface* S, *and if* I *is a boundary curve of* H, *then there exists in* S *a topological image* $\chi(H)$ *of* H *such that the simple closed curve* $\chi(I)$ *constitutes the boundary of one of the components of* $S - \chi(H)$.

In view of Proposition K, let $\chi_\mu(y)$, $\mu = 1, 2$, be a topological transformation of Γ'_μ into a subset $\Gamma''_\mu = \chi_\mu(\Gamma'_\mu)$ of S such that the simple closed curve $\mathbf{T}''_\mu = \chi_\mu(\mathbf{T}'_\mu)$ constitutes the *boundary* of a component I_μ of $S - \Gamma''_\mu$. On setting $h_\mu(P) = \chi_\mu[t_\mu(P)]$, $\mu = 1, 2$, it is clear that the transformation $h(Q) = h_1[h_2^{-1}(Q)]$ is, for $Q \subset \mathbf{T}''_2$, a homeomorphism between the two simple closed curves $\mathbf{T}''_1$ and $\mathbf{T}''_2$. It is furthermore a well known property of a spherical surface that $h(Q)$ can be *extended* to a homeomorphism $H(Q)$ that carries the 2-cell $S - I_2$ into the 2-cell $I_1 + \mathbf{T}''_1$. Now consider the transformation

$$T(P) \begin{cases} = h_1(P), & \text{when} \quad P \subset \Gamma_1, \\ = H[h_2(P)], & \text{when} \quad P \subset \Gamma_2 . \end{cases}$$

[19] It may be observed that $\Pi_{\bar{m}}$ does not necessarily contain a subset homeomorphic with Π unless Π is assumed in the beginning to be homeomorphic with the complex $\mathfrak{C}$ instead of the complex $\mathfrak{L}$. Cf. Introduction.

It is readily seen that $T(P)$ is a topological transformation carrying $\Gamma_1 + \Gamma_2$ into a subset $T(\Gamma_1 + \Gamma_2)$ of the spherical surface S. But the *primitive skew curve* Π is a subset of $\Gamma_1 + \Gamma_2$; consequently we are lead to the contradiction that S itself contains a primitive skew curve, namely, the curve $T(\Pi)$. Having disposed of Case 1, we turn to

Case 2: $\Pi - \mathbf{T}$ contains an *infinite* number of distinct components.

Then if $e_1, e_2, \cdots, e_n, \cdots$ is the set of components of $\Pi - \mathbf{T}$, it is easily seen that $\lim_{n=\infty} \delta(e_n) = 0$. Consider the sequence of continua

$$C_1, C_2, \cdots, C_n, \cdots, \tag{1}$$

where $C_n = \mathbf{T} + \sum_{i=1}^{n} e_i$. Since $\Pi + \mathbf{T} = \mathbf{T} + \sum_{i=1}^{\infty} e_i$ is *not* a continuum of class $\mathfrak{R}$, we conclude by Theorem E_1 that there is some continuum C_k of 1) that *fails* to belong to class $\mathfrak{R}$, and hence contains a *primitive skew curve* Π'. It is now quite evident that the point-set $\Pi' - \mathbf{T}$ contains but a *finite* number of components; consequently Case 2 reduces to Case 1, and our demonstration is completed.

V. The Immersion Theorems

THEOREM I_1. *The continuum* **M** *is homeomorphic with a subset of a peanian continuum of which the proper cyclic elements are simple closed surfaces.*

Demonstration. Consider the Hilbert[20] space $H^\omega = \{(x_1, x_2, \cdots, x_n, \cdots)\}$, and its subspaces

$$H_1, H_2, \cdots, H_n, \cdots, \tag{1}$$

where H_n is defined in H^ω by the equation

$$x_{2n} = x_{2(n+1)} = \cdots = x_{2(n+r)} = \cdots = 0. \tag{2}$$

The spaces H_n are each evidently homeomorphic with H^ω itself. Since the continuum **M** is, *a fortiori*, *compact* and *metric*, it follows from a theorem of Urysohn[21] that **M** is homeomorphic with a subset of H^ω; consequently **M** has a topological image M_1 in H_1.

We introduce a

Spanning Process: Given (1), a continuum M_n of class $\mathfrak{R}$ that lies in H_n, and (2), a *boundary curve* $\mathbf{T}_n$ of M_n; then $\mathbf{T}_n$ is *spanned* with a closed 2-cell $E_n^{(2)}$ in a fashion that

(1) $M_n \cdot E_n^{(2)} = \mathbf{T}_n$,

(2) $M_{n+1}, = M_n + E_n^{(2)}$, is a continuum of class $\mathfrak{R}$ that lies in H_{n+1}, and

(3) $\delta(E_n^{(2)}) \leqq \delta(\mathbf{T}_n) + \frac{1}{n}$.

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[20] F. Hansdorff, Mengenlehre (Berlin, 1927), p. 98.

[21] Über die Metrisation der kompakten topologischen Räume, Math. Ann. 92 (1924), 275–293. Zum Metrisationsproblem, Math. Ann. 94 (1924), 309–315.

The 2-cell $E_n^{(2)}$ is determined as follows: Take an arbitrary point

(4) $p_n, = (x_1^{(n)}, x_2^{(n)}, \cdots, x_{2n-1}^{(n)}, 0, x_{2n+1}^{(n)}, 0, x_{2n+3}^{(n)}, 0, \cdots)$ on $\mathbf{T}_n$, and denote by

(5) P_n the point $(x_1^{(n)}, x_2^{(n)}, \cdots, x_{2n-1}^{(n)}, \frac{1}{n}, x_{2n+1}^{(n)}, 0, x_{2n+3}^{(n)}, 0, \cdots)$ of H_{n+1}.

Then let Z_n be a variable point on $\mathbf{T}_n$, and put

$$E_n^{(2)} = \sum_{Z_n \subset \mathbf{T}_n} P_n Z_n , \tag{6}$$

where $P_n Z_n$ represents, for a given point Z_n, the uniquely determined (closed) *straight interval* in H_{n+1} that is bounded by the two points P_n and Z_n.[22] The point-set $E_n^{(2)}$ is evidently a closed 2-cell lying in H_{n+1} and having $\mathbf{T}_n$ as its *boundary curve* (in our sense as well as in the combinatorial sense). Consequently, since $\mathbf{T}_n$ is the common part of M_n and $E_n^{(2)}$, it follows, by Theorem E_2, that the continuum $M_{n+1} = M_n + E_n^{(2)}$ belongs to class $\mathfrak{R}$ (and is clearly $\subset H_{n+1}$). The relations (4), (5), (6) imply

$$\delta(E_n^{(2)}) = \delta(\mathbf{T}_n + P_n) \leqq \delta(\mathbf{T}_n) + \rho(\mathbf{T}_n, P_n) = \delta(\mathbf{T}_n) + \rho(p_n, P_n) ;$$

and on utilizing

$$\rho(p_n, P_n) = \frac{1}{n}$$

we obtain

$$\delta(E_n^{(2)}) \leqq \delta(\mathbf{T}_n) + \frac{1}{n} .$$

It is seen, therefore, that $E_n^{(2)}$ satisfies the requirements 1), 2), and 3).

Let us assume that the subcontinuum M_1 of H_1 has at least *one* boundary curve; for otherwise, by Theorem 1, our proposition is trivial. Then if d_1 denotes the upper bound of the diameters of the *boundary curves* of M_1, it follows, by Theorem 2, that some such curve, $\mathbf{T}_1$, is itself of diameter d_1. With reference then to M_1 and $\mathbf{T}_1$, apply the spanning process and obtain the continuum $M_2 = M_1 + E_1^{(2)}$. If M_2 also has *boundary curves* let $\mathbf{T}_2$ be one of them

[22] If $P = (x_1, x_2, \cdots, x_n, \cdots)$ and $Q = (y_1, y_2, \cdots, y_n, \cdots)$ are an arbitrary pair of distinct points of H^ω, then we define the *straight interval* PQ to be the set of all points $(z_1, z_2, \cdots, z_n, \cdots)$ of H^ω whose coordinates satisfy the relation

$$d = \frac{z_1 - x_1}{y_1 - x_1} = \frac{z_2 - x_2}{y_2 - x_2} = \cdots = \frac{z_n - x_n}{y_n - x_n} = \cdots ,$$

where $0 \leqq d \leqq 1$.

of maximal diameter d_2 and *span* $\mathbf{T}_2$, obtaining the continuum $M_3 = M_2 + E_2^{(2)}$. By continuing in this manner it is possible that a continuum

$$M_r = M_1 + E_1^{(2)} + \cdots + E_{r-1}^{(2)}$$

will be reached that has *no boundary curves*; in which case our theorem is proved since (by Theorem 1) the proper cyclic elements of M_r are simple closed surfaces and M_r contains the topological image M_1 of $\mathbf{M}$. If, however, on continuing the *spanning process* indefinitely, all the continua M_r, $r \geqq 1$, have *boundary curves*; then consider the sequence of continua

$$M_1, E_1^{(2)}, E_2^{(2)}, \cdots, E_n^{(2)}, \cdots . \tag{7}$$

It has already been observed that

$$M_n = M_1 + \sum_{i=1}^{n-1} E_i^{(2)} \text{ is a continuum of class } \mathfrak{R}; \tag{8}$$

furthermore it is clear that

$$M_1 \cdot E_n^{(2)} \neq 0, \qquad n = 1, 2, 3, \cdots; \tag{9}$$

and we now proceed to establish

$$\lim_{n=\infty} \delta(E_n^{(2)}) = 0. \tag{10}$$

Suppose that (10) does not hold; then from (3) it follows that

$$\lim_{n=\infty} \delta(\mathbf{T}_n) \neq 0.$$

Consequently there exists an increasing sequence of positive integers

$$n_1, n_2, \cdots, n_i, \cdots$$

for which the sequence of *boundary curves*

$$\mathbf{T}_{n1}, \mathbf{T}_{n2}, \cdots, \mathbf{T}_{ni}, \cdots \tag{11}$$

converges to a sequential limit which is of diameter >0 and is hence, by Theorem 2, a boundary curve $\mathbf{T}$ of M_1.[23] Observe that the sequence (11) satisfies the hypothesis of Theorem 3; therefore three integers $u < v < w$ can be chosen sufficiently large that the curves $\mathbf{T}_{nu}$, $\mathbf{T}_{nv}$, $\mathbf{T}_{nw}$ will assuredly have at least *four* distinct common points a, b, c, d which we assume to lie on $\mathbf{T}_{nw}$ in the order $abcd$. Now consider the closed 2-cells $E_{n_u}^{(2)}$ and $E_{n_v}^{(2)}$ that *span* the curves $\mathbf{T}_{nu}$ and $\mathbf{T}_{nv}$ respectively. In $E_{n_u}^{(2)}$ take an arc ac having only its end points a and c on $\mathbf{T}_{nu}$; and similarly in $E_{n_v}^{(2)}$ take an arc bd having only b and d on $\mathbf{T}_{nv}$. The arcs ac and bd evidently have no points in common, and, since the point pairs (a, c) and (b, d) separate each other on $\mathbf{T}_{nw}$, it follows, from Lemma 1, that $\mathbf{T}_{nw}$

[23] That $\mathbf{T}$ is a subset of M_1 follows from $\mathbf{T}_n \subset M_1$ for all values of n.

is not a boundary curve of the continuum M_{n_w}, for the relation $n_w > n_v > n_u$ implies $M_{n_w} \supset E^{(2)}_{n_u} + E^{(2)}_{n_v} \supset ac + bd$. In view of this contradiction, the validity of relation (10) is established.

From (8), (9), and (10) it is now apparent that the sequence of continua (7) satisfies the hypothesis of Theorem E_1; consequently the point set

$$M_\omega = M_1 + \sum_{n=1}^{\infty} E_n^{(2)}$$

is a continuum of class $\mathfrak{R}$. Furthermore, M_ω does *not* have a *boundary curve* for if $\mathbf{T}$ were such a curve, then $\lim_{n=\infty} \delta(\mathbf{T}_n) = 0$ insures $\delta(\mathbf{T}) > \delta(\mathbf{T}_n)$ for n sufficiently large, implying the contradiction that $\mathbf{T}_n$ is not one of the *boundary curves* of maximal diameter d_n of M_n. We conclude, then, from Theorem 1, that the proper cyclic elements of M_ω are simple closed surfaces. Our proposition is therefore proved, since M_ω contains the continuum M_1 which is homeomorphic with $\mathbf{M}$.

The preceding theorem has as an immediate consequence the following corollaries:

COROLLARY 1. *A cyclic continuum of class* $\mathfrak{R}$ *that has precisely one boundary curve is a closed 2-cell.*

This corollary, together with its apparent converse, constitutes another characterization of the closed 2-cell; others having recently been given, independently, by D. W. Woodard,[24] H. Whitney,[25] and L. Zippin.[26]

COROLLARY 2. *Every continuum* $\mathbf{M}$ *of class* $\mathfrak{R}$ *is of Urysohn-Menger dimension* $\leqq 2$.

For, the cyclic elements of $\mathbf{M}$ are, by Theorem I_1, each homeomorphic with a subset of a spherical surface, and hence of dimension $\leqq 2$. Consequently $\mathbf{M}$ itself is of dimension $\leqq 2$.[27]

Let us now consider Theorem I_1 with reference only to the *cyclic* continua of class $\mathfrak{R}$; and observe, then, the interesting result that they are all homeomorphic with a subset of a spherical surface. We have consequently established the sufficiency[28] aspect the following interesting

THEOREM: *A necessary and sufficient condition that a cyclic peanian continuum be homeomorphic with a subset of a spherical surface is that it does not contain a primitive skew curve.*

That this theorem does not hold for peanian continua in general is evident when one considers a point-set consisting of a closed 2-cell $E^{(2)}$ and an arc B

[24] Bull. Amer. Math. Soc., vol. 36 (1930), p. 355.

[25] Tran. Amer. Math. Soc., vol. 35 (1933), pp. 261–273.

[26] Amer. Jour. Math., vol. 55 (1933), pp. 207–217.

[27] C. Kuratowski, Quelques applications d'éléments cycliques de M. Whyburn. **Fund.** Math., vol. 14 (1929), Theorem III, p. 142.

[28] The necessity proof is trivial.

such that $B \cdot E^{(2)}$ is an *interior* point of $E^{(2)}$. Accordingly, we proceed now to determine the *non-cyclic* continua of class $\mathfrak{R}$ that have topological images in the surface of a sphere.

THEOREM I_2. *If the continuum* $\mathbf{M}$ *satisfies the condition* (*L*): *each cut-point* P *of* $\mathbf{M}$ *is a boundary point*[29] *of the closure of every component of* $\mathbf{M} - P$; *then* $\mathbf{M}$ *is homeomorphic with a subset of a spherical surface.*

Demonstration. Suppose that we are given an arbitrary positive number ϵ_n, a non-cyclic continuum M_n of class $\mathfrak{R}$, and a cyclic element C_n of M_n such that

α) M_n is a subset a euclidean six space R_6,

β) M_n satisfies the condition (L),

γ) C_n is *not* an end-point of M_n;

then, we introduce an

Augmentation Process for M_n relative to C_n and ϵ_n: Consider the collection $\mathbf{S}_n$ of *boundary arcs* $P_n^\sigma x_n^\sigma Q_n^\sigma$ of M_n, where $P_n^\sigma x_n^\sigma Q_n^\sigma$ is the sum of its subarcs $P_n^\sigma x_n^\sigma$ and $x_n^\sigma Q_n^\sigma$ in a fashion that

(1) x_n^σ is a cut-point of M_n that is contained in C_n,

(2) $\delta(P_n^\sigma x_n^\sigma) = \delta(x_n^\sigma Q_n^\sigma)$, and

(3) M_n is the sum of two continua X_n^σ and Y_n^σ for which

(i) $X_n^\sigma \supset C_n + P_n^\sigma x_n^\sigma$, $Y_n^\sigma \supset x_n^\sigma Q_n^\sigma$; and

(ii) $X_n^\sigma \cdot Y_n^\sigma = x_n^\sigma$.

The condition (L) insures that the collection $\mathbf{S}_n$ is non-vacuous. Let $\bar{u}_n$ denote the upper bound of the diameters of the arcs of $\mathbf{S}_n$ and take $P_n x_n Q_n$ to be one of them for which

$$\delta(P_n x_n Q_n) > \bar{u}_n/2 . \tag{4}$$

Since M_n is, by Corollary 2, of dimension $\leqq 2$, and consequently, according to the Menger Immersion Theorem,[30] homeomorphic with a subset of euclidean five-space R_5, it follows from a theorem due to Mazurkewicz[31] that M_n is accessible at each of its points from the point-set $R_6 - M_n$. Hence (since M_n

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[29] Cf. Introduction: Definition II.

[30] This theorem, which was enunciated by Menger, is to the effect that every compact n-dimensional space is homeomorphic with a subset of R_{2n+1}. Cf. K. Menger, Dimensionstheorie (Leipzig), 1928, pp. 287–303. Various proofs of this theorem have been given: see papers by G. Nobeling, Math. Annalen, vol. 104 (1930); L. Pontrjagin and G. Tolstowa, Math. Annalen, vol. 105 (1931); and S. Lefschetz, Annals of Math. (2), vol. 32 (1931).

[31] Sur un problem of M. Knaster, Fund. Math., XIII, pp. 146–150.

does not *separate* any region of R_6) there exists an arc $P_n y_n Q_n$ joining the point P_n to the point Q_n such that

$$\overset{\frown}{P_n y_n Q_n} \subset R_6 - M_n , \tag{5}$$

and

$$\delta(P_n y_n Q_n) < \rho(P_n, Q_n) + \epsilon_n . \tag{6}$$

Let us now set

$$M_{n+1} = M_n + P_n y_n Q_n . \tag{7}$$

The point-set M_{n+1} is called an *augmentation set* of M_n relative to C_n and ϵ_n; it is not difficult to show that M_{n+1} is a continuum of class $\mathfrak{R}$ which satisfies the relations $\alpha)$, $\beta)$, and $\gamma)$.

Choose a sequence of positive numbers $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > \cdots$ that converges to zero. Then take in R_6 a topological image M_1 of $\mathbf{M}$, and let C_1 be an arbitrary cyclic element of M_1 that is *not* an end point. With reference to C_1 and ϵ_n, we now apply the augmentation process to M_1 and obtain $M_2 = M_1 + P_1 y_1 Q_1$. If M_2 is *not* cyclic, then let C_2 be *that* one of its cyclic elements which contains $P_1 y_1 Q_1$, and obtain next an augmentation set M_3 $(= M_2 + P_2 y_2 Q_2)$ of M_2 relative to C_2 and ϵ_2. In general, if M_n *fails* to be cyclic, let C_n be *that* one of its cyclic elements which contains the arc $P_{n-1} y_{n-1} Q_{n-1}$, and proceed to obtain the augmentation set M_{n+1} of M_n relative to C_n and ϵ_n.

If, for some value r of n, the continuum M_r is *cyclic*, then our theorem is proved. For $\mathbf{M}$ has a topological image M_1, in M_r; and by Theorem I_1, M_r (being a continuum of class $\mathfrak{R}$) is homeomorphic with a subset of a spherical surface. If, however, for all values of n the continuum M_n has a cut-point, then consider the infinite sequence of continua

$$M_1, P_1 y_1 Q_1, P_2 y_2 Q_2, \cdots , P_n y_n Q_n, \cdots .$$

We have already seen that the continuum $M_n = M_1 + \sum_{i=1}^{n-1} P_i y_i Q_i$ belongs to class $\mathfrak{R}$. It is, furthermore, easy to show

$$M_1 \cdot (P_n y_n Q_n) \neq 0 \qquad n = 1, 2, \cdots ,$$

for as a matter of fact the relations (3), (7), and

$$C_1 \subset C_2 \subset C_3 \cdots \subset C_n \subset \cdots$$

readily imply

$$M_1 \cdot (P_n y_n Q_n) \supset Q_n ,$$

for all values of n. Finally, by a careful consideration of the manner in which the arcs $P_n y_n Q_n$ were chosen (with particular reference to relation (6)), it can be shown that

$$\lim_{n=\infty} \delta(P_n y_n Q_n) = 0 .$$

Therefore, as a result of Theorem E_1, we conclude that the point-set

$$M_\omega = M_1 + \sum_{n=1}^{\infty} P_n y_n Q_n$$

is a continuum of class $\Re$ which, in virtue of relation (4), must also be *cyclic*. Our theorem is consequently established, since $M_1 \subset M_\omega$ and M_ω is homeomorphic with a subset of a spherical surface.

Scarcely any proof is needed in demonstrating the converse of Theorem I_2; and we have now obtained a solution to our original problem that may be stated as follows:

A necessary and sufficient condition that a peanian continuum K *be homeomorphic with a subset of a spherical surface is that*

(1) K *does not contain a primitive skew curve, and*

(2) *each cut-point* P *of* K *is a boundary point of the closure of every component of* $K - P$.

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